More Complete Explanation of the Comparison of Quantities contained in the Integral formula $\int \frac{Zdz}{\sqrt{1+mzz+n^4}}$ while Z denotes any rational function of zz *

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§1 Even though I have treated this subjected already very often and the most illustrious Lagrange published many extraordinary observations on formulas of this kind, it is nevertheless still not to be considered to be sufficiently explored, even less to be exhausted, but still seems to involve many very well hidden secrets, which require a most profound investigation and promises immense increments for this branch of Analysis. But especially these analytic operations, which first led me to this investigation, are of such a nature that they solved the problem only in a very non straight-forward way, whence a direct method leading to the same comparisons is still desired very much. Furthermore, this whole investigation extends a lot further than to the integral formulas, which I contemplated at first, where I assumed only either a constant quantity or a polynomial function of z of this form $F + Gzz + Hz^4 + Iz^6 + Kz^8 +$ etc. for the letter Z, in which cases I showed that having propounded any two quantities of this kind always a third of the same kind can be found,

*Original title: " Plenior Explicatio circa Comparationem Quantitatum in Formula Integrali $\int \frac{Zdz}{\sqrt{1+mzz+n^4}}$ denotante Z functionem quamcunque rationalem ipsius zz", first published in „*Acta academiae scientiarum Petropolitanae*, 1781: II (1785), p.3-22", reprinted in in „*Opera Omnia*: Series 1, Volume 21, pp. 39 - 56 ", Eneström-Number E581, translated by: Alexander Aycock for „Euler-Kreis Mainz" which differs from the sum of those by an algebraic quantity, which certainly vanishes in the case, in which Z is only a constant quantity.

§2 Now, I observed that the same comparisons can be made, if any rational function of zz is taken for Z, which function we certainly want to have a form of this kind

$$\frac{F + Gzz + Hz^4 + Iz^6 + Kz^8 + \text{etc.}}{\mathfrak{F} + \mathfrak{G}zz + \mathfrak{H}z^4 + \mathfrak{I}z^6 + \mathfrak{K}z^8 + \text{etc.}},$$

where this difference occurs, that the difference of the sum of two formulas of this kind and the third formula of the same kind to be found no longer is an algebraic quantity, but can nevertheless be exhibited by means of logarithms and circular arcs, such that now the investigation extends a lot further than I had covered it up to this point. And hence, if all operations, which lead to this goal, are considered with the appropriate attention, they will maybe be able to open a simpler way to get to a direct method and to persecute this whole most mysterious subject with higher success.

§3 But that all these things can be seen more clearly, let this character $\Pi : z$ denote the transcendental function, which results from the integration of the propounded formula

$$\int \frac{Zdz}{\sqrt{1 + mzz + nz^4}},$$

where the integral is assumed to be taken in such a way that it vanishes for $z = 0$; hence it is immediately obvious that it will also be $\Pi : 0 = 0$. Further, because Z involves only even powers of z, and likewise also only even powers are contained in the square root, it is evident, if one writes $-z$ instead of z, that then the value of this integral formula and hence also of this character $\Pi : z$ goes over into its negative, such that it is $\Pi : (-z) = -\Pi : z$. Having noted these things in advance, if any two quantities of this kind $\Pi : p$ and $\Pi : q$ are propounded, it is always possible to find a third quantity of the same kind $\Pi : r$, which differs from the sum of those formulas $\Pi : p + \Pi : q$ by an either algebraic quantity or at least a quantity assignable by means of logarithms and circular arcs. But the rule, by which from the given letters p and q the third is found, always remains the same, no matter which function is denoted by the letter Z; for, it will always be

$$r = \frac{p\sqrt{1+mqq+nq^4}+q\sqrt{1+mpp+np^4}}{1-nppqq}$$

But hence for the following combinations it will be helpful to have observed that it will be

$$\sqrt{1+mrr+n^r}$$
$$= \frac{(mpq+\sqrt{1+mpp+np^4}\sqrt{1+mqq+nq^4})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

§4 But this investigation is not restricted only to formulas of this kind $\Pi : p$ and $\Pi : q$ to be taken arbitrarily, but can also be extended to any arbitrary number of given formulas, such that, no matter how many formulas of this kind were propounded, of course

$$\Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.},$$

always a new formula of this kind $\Pi : r$ can be assigned, which differs from the sum of those either by an algebraic quantity or at least one assignable by logarithms and by circular arcs. It will even be possible to define those formulas, which we considered as given, in such a way that this either algebraic difference or difference depending on logarithms and circular arcs vanishes completely, such that it will be

$$\Pi : r = \Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

And these are almost the things to which I could extend this more general investigation, which I decided to explain here; therefore, I will succinctly present the single operations, which led me to these results.

Operation 1

§5 I started this whole investigation from the consideration of this algebraic equation

$$\alpha + \gamma(xx+yy) + 2\delta xy + \zeta xxyy = 0,$$

from which, since it is quadratic, by extracting the roots so for x as for y one concludes either

$$y = \frac{-\delta x + \sqrt{-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4}}{\gamma + \zeta xx}.$$

or

$$x = \frac{-\delta y + \sqrt{-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4}}{\gamma + \zeta yy}.$$

such that hence it is

$$\sqrt{-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4} = \gamma y + \delta x + \zeta xxy$$

and

$$\sqrt{-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4} = \gamma x + \delta y + \zeta xyy.$$

§6 Now I define the letters α, γ, δ, ζ etc. in such a way that both formulas are reduced to the form

$$\sqrt{1 + mxx + nx^4} \quad \text{and} \quad \sqrt{1 + myy + ny^4},$$

for which aim I put

$$1. \quad -\alpha\gamma = k, \quad 2. \quad \delta\delta - \gamma\gamma - \alpha\zeta = mk \quad \text{and} \quad 3. \quad -\gamma\zeta = nk;$$

from the first of these equations it is $\alpha = -\frac{k}{\gamma}$, from the third $\zeta = \frac{-nk}{\gamma}$, which value substituted in the second yields

$$\delta\delta = \gamma\gamma + \frac{nkk}{\gamma\gamma} + mk$$

and hence

$$\delta = \sqrt{\gamma\gamma + \frac{nkk}{\gamma\gamma} + mk} = \frac{1}{\gamma}\sqrt{\gamma^4 + m\gamma\gamma k + nkk},$$

whence our equation will now be

$$-k + \gamma(xx + yy) + 2xy\sqrt{\gamma^4 + m\gamma\gamma k + nkk} - nkxxyy = 0;$$

therefore, hence both our irrational formulas will be

$$\sqrt{k(1+mxx+nx^4)} = \gamma y + \frac{1}{\gamma}x\sqrt{\gamma^4+m\gamma\gamma k+nkk} - \frac{nk}{\gamma}xxy,$$
$$\sqrt{k(1+myy+ny^4)} = \gamma x + \frac{1}{\gamma}y\sqrt{\gamma^4+m\gamma\gamma k+nkk} - \frac{nk}{\gamma}xyy.$$

§7 Since now the two quantities x and y depend on each other in such a way, as the assumed equation declares it, let us define the still undefined γ and k in such a way that for $x=0$ it is $y=a$. Therefore, it has to be $-k+\gamma\gamma aa=0$ and hence $k=\gamma\gamma aa$; having substituted this value our equation will be

$$0 = \gamma\gamma(xx+yy-aa) + 2\gamma\gamma xy\sqrt{1+maa+na^4} - n\gamma\gamma aaxxyy,$$

and hence by diving by $\gamma\gamma$ it will be

$$0 = (xx+yy-aa) + 2xy\sqrt{1+maa+na^4} - naaxxyy.$$

But then our two formulas containing the roots will be expressed this way

$$\sqrt{1+mxx+nx^4} = \frac{y}{a} + \frac{x}{a}\sqrt{1+maa+na^4} - naxxy,$$
$$\sqrt{1+myy+ny^4} = \frac{x}{a} + \frac{y}{a}\sqrt{1+maa+na^4} - naxyy.$$

§8 To render these formulas more tractable, let us put

$$\sqrt{1+maa+n^4} = \mathfrak{A}$$

and in like manner

$$\sqrt{1+mxx+nx^4} = \mathfrak{X} \quad \text{and} \quad \sqrt{1+myy+ny^4} = \mathfrak{Y}$$

and our equation will be

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

whence one finds

$$y = -\frac{\mathfrak{A}x - a\mathfrak{X}}{1-naaxx} \quad \text{and} \quad x = -\frac{\mathfrak{A}y - a\mathfrak{Y}}{1-naayy};$$

hence it is plain, if it was $y = 0$, that it will be $x = a$; but then the formulas involving the square roots will be

$$\begin{aligned}
\sqrt{1 + mxx + nx^4} &= \mathfrak{X} = \frac{y}{a} + \frac{\mathfrak{A}x}{a} - naxxy, \\
\sqrt{1 + myy + ny^4} &= \mathfrak{Y} = \frac{x}{a} + \frac{\mathfrak{A}y}{a} - naxyy.
\end{aligned}$$

§9 But as it was possible to express both y by x and x by y, so it will also be possible to express $\mathfrak{Y}$ only in terms of x and $\mathfrak{X}$ only in terms of y. But after the calculation it will be seen that it will be

$$\begin{aligned}
\mathfrak{X} &= \frac{(-may + \mathfrak{A}\mathfrak{Y}(1 + naayy) - 2nay(aa + yy)}{(1 - naayy)^2}, \\
\mathfrak{Y} &= \frac{(-max + \mathfrak{A}\mathfrak{X}(1 + naaxx) - 2nax(aa + xx)}{(1 - naaxx)^2}.
\end{aligned}$$

§10 But it especially deserves to be noted on our equation

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

that the three quantities xx, yy, aa are completely interchangeable. For, if the irrational term is brought to the other side that it is

$$xx + yy - aa - naaxxyy = -2\mathfrak{A}xy,$$

and the squares are taken, by substituting its value $1 + maa + na^4$ for $\mathfrak{A}^2$ again this equation will result

$$\left.\begin{array}{llll}
+\,x^4 & -\,2xxyy & -\,4maaxxyy & -\,2na^4xxyy + nna^4x^4y^4 \\
+\,y^4 & -\,2aaxx & & -\,2naax^4yy \\
+\,a^4 & -\,2aayy & & -\,2naaxxy^4
\end{array}\right\} = 0,$$

where the interchangeability of the letters a, x, y immediately catches the eye. In the above formulas, where the quantity a itself enters, the interchangeability is not that obvious, but is completely clear, if we write $-b$ instead of $-a$ and $\mathfrak{B}$ instead of $\mathfrak{A}$; for, then, as it was

$$y = -\frac{x\mathfrak{B} + b\mathfrak{X}}{1 - nbbxx} \quad \text{and} \quad x = -\frac{y\mathfrak{B} + b\mathfrak{Y}}{1 - nbby},$$

so it will be

$$b = -\frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$$

and in like manner for the formulas containing the square roots or the small letters it will be

$$\begin{aligned}
\mathfrak{Y} &= \frac{(mbx + \mathfrak{B}\mathfrak{X})(1 + nbbxx) + 2nbx(bb + xx)}{(1 - nbbxx)^2}, \\
\mathfrak{X} &= \frac{(mby + \mathfrak{B}\mathfrak{Y})(1 + nbbyy) + 2nby(bb + yy)}{(1 - nbbyy)^2}, \\
\mathfrak{B} &= \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}
\end{aligned}$$

and so the complete interchangeability is seen.

Operation 2

§11 Now, let us differentiate our assumed algebraic equation, which is

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

and the differential equation will be

$$dx(x + \mathfrak{A}y - naayxx) + dy(y + \mathfrak{A}x - naaxxy) = 0$$

or

$$\frac{dx}{y + \mathfrak{A}x - naaxxy} + \frac{dy}{x + \mathfrak{A}y - naaxyy} = 0.$$

But from the above expressions it is known to be

$$y + \mathfrak{A}x - naaxxy = a\mathfrak{X} \quad \text{and} \quad x + \mathfrak{A}y - naaxyy = a\mathfrak{Y},$$

whence the differential equation will obtain this form

$$\frac{dx}{a\mathfrak{X}} + \frac{dy}{a\mathfrak{Y}} = 0$$

or

$$\frac{dx}{\sqrt{1+mxx+nx^4}} + \frac{dy}{\sqrt{1+myy+ny^4}} = 0.$$

§12 Therefore, having found this differential equation let this character $\Gamma : x$ denote the integral $\int \frac{dx}{\mathfrak{X}}$ and the character $\Gamma : y$ the integral $\int \frac{dy}{\mathfrak{Y}}$ having taken the integral in such a way that it vanishes for $x = 0$ or $y = 0$, and that differential equation by integrating it will become $\Gamma : x + \Gamma : y = C$. But because having taken $x = 0$ it also is $\Gamma : x = 0$ and $y = a$, that constant will be $C = \Gamma : a$, such that we have this equation $\Gamma : x + \Gamma : y = \Gamma : a$.

§13 Since here no other variability is taken into account, it is plain having taken the two letters the x and y arbitrarily that a can always be defined in such a way that it is

$$\Gamma : a = \Gamma : x + \Gamma : y.$$

For, if in § 10 one writes $-a$ instead of b, one has to take

$$a = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy},$$

which comparison now constitutes a special case of the general investigation we are doing. For, if we write p and q instead of x and y, but r instead of a, but then $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{R}$ instead of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{A}$ and if, having taken the quantities p, q arbitrarily, one takes $r = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$, then it will certainly be $\Gamma : r = \Gamma : p + \Gamma : q$, such that in this case that difference of $\Gamma : r$ and the sum $\Gamma : p + \Gamma : q$ vanishes completely. And so we have already expanded the case in which our general form

$$\int \frac{Zdz}{\sqrt{1+mzz+nz^4}}$$

a constant quantity is taken for Z.

OPERATION 3

§14 Now, to get closer to our actual goal, let X and Y be such functions of x and y, as we want Z to be one of z, and since we just found

$$\frac{dx}{\sqrt{1+mxx+nx^4}}+\frac{dy}{\sqrt{1+myy+ny^4}}=0,$$

let us put that it is

$$\frac{Xdx}{\sqrt{1+mxx+nx^4}}+\frac{Ydy}{\sqrt{1+myy+ny^4}}=dV,$$

such that, if X and Y were constant quantities, it would be $dV=0$, Therefore, hence, if we write $\frac{-dx}{\sqrt{1+mxx+nx^4}}$ instead of $\frac{dy}{\sqrt{1+myy+ny^4}}$, it would be

$$dV=\frac{(X-Y)dx}{\sqrt{1+mxx+nx^4}} \quad \text{or also} \quad dV=\frac{(Y-X)dy}{\sqrt{1+myy+ny^4}}.$$

But if we write its rational values instead of the roots, it will be

$$dV=\frac{a(X-Y)dx}{y+\mathfrak{A}x-naaxxy} \quad \text{or} \quad dV=\frac{a(Y-X)dy}{x+\mathfrak{A}y-naaxyy}.$$

§15 But because there is no reason, why we should express this differential dV rather in terms of dx than in terms of dy, it will be wise to introduce a new quantity into the calculation, which is equally related to x and y. For this purpose, let us set the product $xy=u$ and put

$$\frac{dx}{y+\mathfrak{A}x-naaxxy}=\frac{dy}{x+\mathfrak{A}y-naaxyy}=sdu.$$

Therefore, it will hence be

$$dx=sdu(y+\mathfrak{A}x-naaxxy) \quad \text{and} \quad dy=-sdu(x+\mathfrak{A}y-naaxyy),$$

whence we conclude

$$ydx+xdy=sdu(yy-xx)=du,$$

and so we will have $s=\frac{1}{yy-xx}$ such that we have

$$\frac{dx}{y+\mathfrak{A}x-naaxxy}=-\frac{dy}{x+\mathfrak{A}y-naaxyy}=\frac{du}{yy-xx}.$$

Therefore, having substituted these values we obtain

$$dV=\frac{a(X-Y)du}{yy-xx}=\frac{-adu(X-Y)}{xx-yy}.$$

§16 But because X and Y are even rational functions of x and y, in which only the even powers of these letters are contained, it is easily seen that the formula $X-Y$ is always divisible by $xx-yy$ and the quotient except for the product $xy=u$ additionally involves the sum of the squares $xx+yy$; therefore, let us set $xx+yy=t$, and because our fundamental equation becomes

$$t-aa+2\mathfrak{A}u-naauu=0,$$

from it it is

$$t=aa-2\mathfrak{A}u+naauu,$$

such that t becomes equal to a rational function of u. Therefore, if this value is written instead of t, our differential in question dV will be expressed only by the variable u, such that having put $dV=udU$ U always is a rational function of u; therefore, if it was polynomial, then V will become equal to an algebraic function of u, but if it is a fractional function, then the integral $V=\int Udu$ can always be exhibited by means of logarithms and circular arcs. Therefore, if this integral is taken in such a way that it vanishes for $u=xy=0$, the same will also vanish for $x=0$ or $y=0$. And hence by integration we will obtain

$$\int\frac{Xdx}{\sqrt{1+mxx+nx^4}}+\int\frac{Ydy}{\sqrt{1+myy+ny^4}}=C+V=C+\int Udu.$$

§17 Therefore, if the characters $\Pi:x$ and $\Pi:y$ denote the values of these integrals, such that both of them vanish having taken either $x=0$ or $y=0$, since for $x=0$ by assumption it is $y=a$, it is obvious that this constant will be $\Pi:a$ and so this finite equation will result

$$\Pi:x+\Pi:y=\Pi:a+\int Udu.$$

§18 But let us investigate the values of this fraction U for a certain case more accurately. And at first, if one takes

$$Z = \alpha + \beta zz + \gamma z^4 + \delta z^6 + \text{etc.},$$

it will in like manner be

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{etc.} \quad \text{and} \quad Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.};$$

hence, because we found

$$dV = Udu = -\frac{adu(X-Y)}{xx-yy},$$

it will be

$$U = -\frac{a(X_Y)}{xx-yy} \quad \text{and hence} \quad U = -\frac{a(xx-yy) + \gamma(x^4-y^4) + \delta(x^6-y^6)}{xx-yy},$$

whence it is

$$U = -a\beta - a\gamma(xx+yy) - a\delta(x^4 + xxyy + y^4).$$

Therefore, because it is $xx + yy = t$ and $xy = u$, it will be

$$U = -a\beta - a\gamma t - a\delta(tt - uu);$$

hence, because it is $t = aa - 2\mathfrak{A}u + naauu$, having done the calculation it will be

$$\int Udu = -a\beta u - a\gamma\left(aau - \mathfrak{A}uu + \frac{1}{3}naau^3\right)$$
$$-a\delta\left(a^4u - 2aa\mathfrak{A}uu + \frac{2}{3}na^4u^3 + \frac{4}{3}\mathfrak{A}^2u^3 - \frac{1}{3}u^3 - n\mathfrak{A}a^2u^4 + \frac{1}{5}n^2a^4u^5\right).$$

And hence it is understood, if the function Z rises to higher powers, how the value of the integral of $\int Udu$ can hence be found.

§19 But if Z was a rational function, of course

$$Z = \frac{\alpha + \beta zz + \gamma z^4}{\zeta + \eta zz + \theta z^4}$$

and hence

$$X = \frac{\alpha + \beta xx + \gamma x^4}{\zeta + \eta xx + \theta x^4} \quad \text{and} \quad Y = \frac{\alpha + \beta yy + \gamma y^4}{\zeta + \eta yy + \theta y^4}.$$

it will be

$$X - Y = \frac{(\beta\zeta - \alpha - \eta)(xx - yy) + (\gamma\zeta - \alpha\theta)(x^4 - y^4) + (\gamma\eta - \beta\theta)x^2y^2(x^2 - y^2)}{\zeta\zeta + \zeta\eta(xx + yy) + \zeta\theta(x^4 + y^4) + \eta^2x^2y^2 + \eta\theta x^2y^2(xx + yy) + \theta\theta x^4y^4}.$$

Therefore, hence having introduced the letters t and u it will be

$$\frac{X - Y}{xx - yy} = \frac{\beta\zeta - \alpha\eta + (\gamma\zeta - \alpha\theta)t + (\gamma\eta - \beta\theta)uu}{\zeta\zeta + \zeta\theta t(tt - 2uu) + \eta\eta uu + \eta\theta tuu + \theta\theta u^4};$$

therefore, because it is

$$U = -\frac{a(X - Y)}{xx - yy},$$

because of $t = aa - 2\mathfrak{A}u + naauu$ it is obvious that the integral of the formula $\int Udu$, if it is not algebraic, can always be exhibited by logarithms or circular arcs. And so by means of these three operations we achieved everything, which is necessary to solve all questions extending to this subject.

Problem 1

§20 *If $\Pi : x$ and $\Pi : y$ denote the values of the integral formulas*

$$\int \frac{Xdx}{\sqrt{1 + mxx + nx^4}} \quad \textit{and} \quad \int \frac{Ydy}{\sqrt{1 + myy + ny^4}},$$

where X and Y are the same even functions of x and y, and two formulas of this kind $\Pi : x$ and $\Pi : y$ are given, to find a third formula of the same kind $\Pi : z$ that it is $\Pi : z = \Pi : x + \Pi : y + W$ such that W is either an algebraic function or a function assignable by logarithms and circular arcs.

SOLUTION

Because the two quantities x and y are given, from them form the square roots

$$\mathfrak{X} = \sqrt{1 + mxx + nx^4} \quad \text{and} \quad \mathfrak{Y} = \sqrt{1 + myy + ny^4};$$

and from them define the quantity z in the same way we taught to define the letter a by x and y, such that it is

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$$

and its irrational value

$$\mathfrak{Z} = \sqrt{1 + mzz + nz^4} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2};$$

then in the above formulas let us write z instead of a everywhere and take $U = -\frac{z(X-Y)}{xx-yy}$, which quantity we saw that it can always be reduced to a function of u while $u = xy$, and put $V = \int Udu$, in which integration the quantities z and $\mathfrak{Z}$ are to be considered as constant, such that the letter V can be considered as a function of $u = xy$, since also z and $\mathfrak{Z}$ are determined by x and y. But note that in this integral formula only the quantity u is to be treated as a variable. Therefore, having found the quantity V it will be

$$\Pi : x + \Pi : y = \Pi : z + V;$$

hence, because it must be

$$\Pi : z = \Pi : x + \Pi : y + W,$$

it is plain that it is $W = -V$ and hence either an algebraic quantity or one assignable by logarithms or circular arcs.

COROLLARY 1

§21 Therefore, the whole task here reduces to the integration of the formula UdU while $u = xy$ and $U = -\frac{z(X-Y)}{xx-yy}$, which we saw above that it can always be expressed by u, if in this integration the letters z and $\mathfrak{Z}$ are treated as constant quantities.

Corollary 2

§22 Therefore, because for a given form of the two functions X and Y the integration is done without any difficulty and the integral itself is always expressed by u, this means by xy, whose value can always be exhibited from the given quantities x and y, in the following we will write the character $\Phi : xy$ instead of the quantity Y, whence for any other letters assumed instead of x and y the meaning of the characters $\Phi : pq$, $\Phi : ab$ etc. are understood.

Corollary 3

§23 Therefore, using this character, if we take $Z = \frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$ for the given quantities x and y, whence it is

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}$$

it will be

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Problem 2

§24 *Using the all characters we explained up to now, if three formulas $\Pi : p$, $\Pi : q$, $\Pi : r$ are given, to find a fourth of the same kind $\Pi : z$ that it was*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + W,$$

such that W is an algebraic quantity or one assignable by logarithms or circular arcs.

Solution

From the two given quantities p and q and hence also $\mathfrak{P}$ and $\mathfrak{Q}$ which have to result from them take

$$x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$$

and at the same time

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

But then also calculate the value of the character $\Phi : pq$ and by means of the preceding formulas it will be

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq$$

or

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq;$$

having substituted this value it will be

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + W.$$

But from the preceding problem, if we write r instead of y here and take

$$z = \frac{x\mathfrak{R} + r\mathfrak{X}}{1 - nrrxx},$$

whence it is

$$\mathfrak{Z} = \frac{(mrx + \mathfrak{R}\mathfrak{X})(1 + nrrxx) + 2nrx(rr + xx)}{(1 - nrrxx)^2},$$

it will be

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx,$$

having combined which form with the preceding one this expression is concluded

$$W = -\Phi : pq - \Phi : rx,$$

such that it is

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx.$$

PROBLEM 3

§25 *Having propounded four formulas of this kind* $\Pi : p$, $\Pi : q$, $\Pi : r$, $\Pi : s$ *to find a fifth of the same kind* $\Pi : z$ *such that it is*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s + W,$$

such that W *is an algebraic quantity or a quantity assignable by logarithms or circular arcs.*

SOLUTION

From the given two letters p and q find x that it is

$$x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq},$$

so

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2},$$

and it will be

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq.$$

In like manner, from the given two r and s find y that it is

$$y = \frac{r\mathfrak{S} + s\mathfrak{R}}{1 - nrrss},$$

and it will be

$$\mathfrak{Y} = \frac{(mrs + \mathfrak{R}\mathfrak{S})(1 + nrrss) + 2nrs(rr + ss)}{(1 - nrrss)^2},$$

but then

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs.$$

Now finally, from the found x and y take

$$z = \frac{x\mathfrak{Y}}{1 - nxxyy} \quad \text{and} \quad \mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}$$

and it will be

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Therefore, if the values just found are substituted for $\Pi : x$ and $\Pi : y$, it will be

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : pq - \Phi : rs - \Phi : xy.$$

COROLLARY 1

§26 Hence it is clearly understood, if an arbitrary number of formulas of this kind is propounded, how a new one of the same kind $\Pi : z$ must be investigated, which differs from those, if they are added, by an algebraic quantity or a quantity assignable by logarithms or circular arcs.

COROLLARY 2

§27 If all those formulas were equal to each other and their number is $= \lambda$, one will always be able to find a new formula $\Pi : z$ that it is

$$\Pi : z = \lambda \Pi : p + W$$

while W is either an algebraic quantity or one assignable by logarithms or circular arcs. Having propounded the formulas $\Pi : p$ and $\Pi : q$ one will even be able to find $\Pi : z$ that it is

$$\Pi : z = \lambda \Pi : p + \mu \Pi : q + W.$$

SCHOLIUM

§28 Therefore, this way I think to have explained not only the principles and fundamentals, on which this subject is based, succinctly and clearly, but also developed the theory a lot more than it was done until now. But a more direct way, which leads to the same investigations, is still desired very much. For, certainly nobody will doubt that hence very large progress in whole analysis will follow.

APPLICATION TO TRANSCENDENTAL QUANTITIES CONTAINED IN THE FORM $\int \frac{dz(\alpha+\beta zz)}{\sqrt{1+mzz+nz^4}}$

§29 Therefore, because here it is $Z = \alpha + \beta zz$, having propounded two formulas of the kind $\Pi : x$ and $\Pi : y$ and having taken

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy} \quad \text{and hence} \quad \mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(x^2 + y^2)}{(1 - nxxyy)^2}$$

from § 18, where it is $u = xy$ and $a = z$, it will be

$$\Pi : z = \Pi : x + \Pi : y + \beta xyz,$$

such that the character used before $\Phi : xy$ in this case obtains the value βxyz. Therefore, by means of this rule having propounded two formulas of this kind $\Pi : x$ and $\Pi : y$ one can always find a third $\Pi : z$ which differs from the sum of those by an algebraic quantity.

§30 Therefore, let us put that any arbitrary number of transcendental formulas of this kind is propounded

$$\Pi : a, \quad \Pi : b, \quad \Pi : c, \quad \Pi : d, \quad \Pi : e, \quad \Pi : f, \quad \Pi : g \quad \text{etc.}$$

and from the single quantities a, b, c, d etc. the irrational values denoted by the Germanic letters are calculated to be

$$\begin{aligned} \mathfrak{A} &= \sqrt{1 + maa + na^4}, & \mathfrak{B} &= \sqrt{1 + mbb + nb^4}, \\ \mathfrak{C} &= \sqrt{1 + mcc + nc^4}, & \mathfrak{D} &= \sqrt{1 + mdd + nd^4}, \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

then one will always be able to exhibit a formula of the same kind, which differs from their sum by an algebraic quantity, no matter how large the number of the given formulas was. But the operations leading to this goal will be illustrated most conveniently the following way.

§31 At first, from two of the given letters a and b find p that it is

$$p = \frac{a\mathfrak{B} + b\mathfrak{A}}{1 - naabb} \quad \text{and} \quad \mathfrak{P} = \frac{(mab + \mathfrak{A}\mathfrak{B})(1 + naabb) + 2nab(aa + bb)}{(1 - naabb)^2}.$$

Further, from this quantity p together with a third of the given ones, c, define q that it is

$$q = \frac{p\mathfrak{C} + e\mathfrak{P}}{1 - nccpp} \quad \text{and} \quad \mathfrak{Q} = \frac{(mcp + \mathfrak{C}\mathfrak{P})(1 + ccpp) + 2nacp(cc + pp)}{(1 - nccpp)^2}.$$

Thirdly, from this quantity q together with the fourth of the given ones, d, find r that it is

$$r = \frac{q\mathfrak{D} + d\mathfrak{Q}}{1 - nddqq} \quad \text{and} \quad \mathfrak{R} = \frac{(mdq + \mathfrak{D}\mathfrak{Q})(1 + nddqq) + 2ndq(dd + qq)}{(1 - nddqq)^2}.$$

Fourthly, from this quantity r together with the fifth of the given ones, e, define s that it is

$$s = \frac{r\mathfrak{E} + e\mathfrak{R}}{1 - neerr} \quad \text{and} \quad \mathfrak{S} = \frac{(mer + \mathfrak{E}\mathfrak{R})(1 + neerr) + 2ner(ee + rr)}{(1 - neerr)^2}.$$

And continue these operations, until all given quantities were introduced into the calculation.

§32 But having found all these values the following comparisons listed in order will be

$$\begin{aligned}
&\text{I.} \quad \Pi : p = \Pi : a + \Pi : b + \beta abp,\\
&\text{II.} \quad \Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq,\\
&\text{III.} \quad \Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr,\\
&\text{IV.} \quad \Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e\\
&\qquad + \beta abp + \beta cpq + \beta dqr + \beta ers,\\
&\text{V.} \quad \Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f\\
&\qquad + \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst\\
&\text{etc.}
\end{aligned}$$

§33 Therefore, because this transcendental formula

$$\Pi : z = \int \frac{dz(\alpha + +\beta zz)}{\sqrt{1 + mzz + nz^4}}$$

contains the arcs of all conic sections beginning at the vertex, by means of these formulas, no matter how many arcs on certain conic section are propounded, which all begin at the vertex, one will always be able separate a new arc equally beginning at the vertex on the same conic section; and this arc then differs from the sum of those given arcs by an algebraically assignable quantity. There is even no obstruction that some of the given arcs are taken negatively, since we already noted that it is $\Pi : (-z) = -\Pi : z$, such that our determination can also be accommodated to arcs within any given boundaries. And this way the treatment, which I gave recently for the comparison of such arcs, can be generalized a lot.

§34 Furthermore, as in this case, in which we took $Z = \alpha + \beta zz$, the character used above $\Phi : xy$ went over into βxyz, while from the two quantities x and y according to the given prescriptions the third z is determined, so, whatever other function is used instead of Z, since we put

$$\Phi : xy = -a \int \frac{(X - Y)du}{xx - yy}$$

with $u = xy$, having done the integration the function resulting from this will also contain the quantity u together with the letters a and $\mathfrak{A}$, since the letter t was expressed in such a way

$$t = aa - 2\mathfrak{A}u + naauu,$$

if, having found the integral, one writes xy instead of u everywhere, but the letters z and $\mathfrak{Z}$ instead of a and $\mathfrak{A}$; and this way the value of the character $\Phi : xy$ is obtained for any propounded case, which function, if it was not algebraic, can always be exhibited by logarithms and circular arcs, if, as we assumed, the letter Z was a rational even function of z.